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Title: Comments on noncommutative quantum mechanical systems associated with Lie algebras

We consider quantum mechanics on the noncommutative spaces characterized by the commutation relations

$$ [x_a, x_b] \ =\ i\theta f_{abc} x_c\,, $$ where f_{abc} are the structure constants of a Lie algebra. We note that the quantum problems in this noncommutative setting can be reformulated as ordinary quantum problems in the commuting {\it momentum} space. The coordinates are then represented as linear differential operators $\hat x_a = iE_{ab} (p)\, \partial /\partial p_b $. Generically, the matrix $E_{ab}(p)$ represents a certain infinite series over the deformation parameter θ: $E_{ab} = \delta_{ab} + \ldots$.

For semisimple compact Lie algebras, the naturally chosen Hamiltonian, $\hat H \ =\ \frac 12 \hat x_a^2\,, $ coincides with the Laplace-Beltrami operator describing the motion along the corresponding group manifold endowed by the metric invariant under left and right group rotations. Then E_{ab} have the meaning of vielbeins. The characteristic size of the manifold is of order θ^{-1}.

For the algebras $u(N)$, the operators $\hat x_a$ can be represented in a simple finite form with only two terms in the expansion in θ. When $N=2$, this gives rise to the Hermitian Hamiltonian describing the motion along a non-compact cover of $U(2)$. When $N \geq 3$, such representation involves a non-Hermitian Hamiltonian.

A byproduct of our study are new nonstandard formulas for the metrics on all the spheres S^n, on the corresponding projective spaces RP^n and on the cover of $U(2)$.